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Solitary waves in trapezoidal channels

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Experimental measurements were made of the variation of water-surface height across the crests of solitary waves travelling along trapezoidal channels. The channels had one vertical side and the other side sloped at either 30°, 45° or 60° to the vertical. Reliable results were not obtained with the last channel, but for the other channels there was reasonable agreement with recent theoretical results.

1. Introduction

Recent theoretical work has given a detailed description of long gravity waves on water in straight uniform channels of arbitrary cross-section. This paper reports some simple experiments designed to compare one aspect of the theoretical results with actual waves. Peters (1966) developed the theory for steady translational waves, such as a solitary wave, including an arbitrary initial distribution of velocity over the cross-section of the channel. Peregrine (1968) approached the problem from a slightly different view-point and developed equations for unsteady wave motion on otherwise still water. These equations give steady waves in agreement with Peters's results.

Both authors take linearized long waves as the first approximation. To this approximation the pressure is hydrostatic, the water surface elevation does not vary across the channel, and the longitudinal velocity is uniform over the cross-section of the channel. The next approximation includes quadratic terms in the equations of motion, and also includes the effects of vertical and transverse accelerations of the water on the pressure. This results in a variation of the surface elevation and longitudinal velocity across the channel.

It is the variation of surface elevation across the channel at the crest of a solitary wave that was measured in the experiments. They were in trapezoidal channels with one side vertical and the other sloping at either 30°, 45° or 60° to the vertical. No good results were obtained in the latter case, but quite reasonable agreement with the theory was obtained for channels with sides at 30° and 45° to the vertical except for one particular situation. The velocity of the waves was also measured but the accuracy of measurement (1 to 2 %) was insufficient to confirm the theoretical values, though the measurements were consistent with the theory.

Sandover & Taylor (1962) performed a number of experiments with undular bores in trapezoidal channels. In a rectangular channel the first wave of an undular bore is very closely similar to a solitary wave provided it does not break.

It is reasonable to assume that the same is true of the first wave of an undular bore in a trapezoidal channel, especially since the equations of motion in the two cases are similar. The velocities of undular bores in trapezoidal channels given by Sandover & Taylor in figure 10 of their paper are mostly about 2 or 3 % greater than the theoretical value for a solitary wave. However, only a few of them had a small amplitude so that very good agreement cannot be expected. They also measured the transverse profile of the wave crest, but their results in figure 9 (of their paper) show a kink in the profile in many cases. If the edge of the channel has not been previously wetted, it is possible that such a kink may be due to a ripple coming from the front of the wave.

2. Theory

The details of the theory were given in Peregrine (1968), who showed, in his equation (10), that the variation of crest height across the channel is given in non-dimensional form by

$$\frac{\partial^2 u}{\partial x\,\partial t}\psi(y,0).$$

The axes are chosen with Ox along the channel and Oz along the vertical, and a typical depth h_0 and gravity are used to make the variables non-dimensional. u is the component of water velocity along the channel and is uniform over any cross-section to the first approximation. The function $\psi(y,z)$ satisfies

$$\frac{\partial^2 \psi}{\partial y^2}+\frac{\partial^2 \psi}{\partial z^2}=1$$

in the cross-section of the channel with boundary conditions

$$\partial\psi/\partial n = 0 \quad \text{on solid boundaries,}$$

$$\partial\psi/\partial z = A_0/B_0 \quad \text{at } z=0\text{, the undisturbed free surface.}$$

A_0 is the undisturbed cross-sectional area of the channel and B_0 is its width at $z = 0$.

For the trapezoidal channel the depth h_0 was taken equal to the maximum depth of the channel. The width at the bottom was taken to be L and the angle of the sloping side to the vertical to be α (figure 1). The function $\psi(y,z)$ was computed numerically using the extrapolated Liebmann method (successive over-relaxation). An example of a solution, for $L = 2, \alpha = 45°$, is given in figure 2. The transverse motion of the water is perpendicular to lines of constant ψ. There is an arbitrary constant to be determined in $\psi(y,z)$ which corresponds to that part of the second-order approximation to the amplitude which is constant across the channel. However, only the first approximation to the amplitude is required in finding the variation in height across the channel as given above. The first approximation to the amplitude was taken to be the amplitude next to the vertical side, and ψ was taken to be zero there, as in figure 2.

It was assumed that the waves photographed were indeed solitary waves so that once their amplitude had been measured the theoretical height variation across the channel could be worked out. If a is the amplitude, the height variation is

$$\frac{a^2B_0(1-\frac{1}{3}B'(0)A_0/B_0^2)\psi(y,0)}{2A_0(\psi_B-\psi_A)},$$

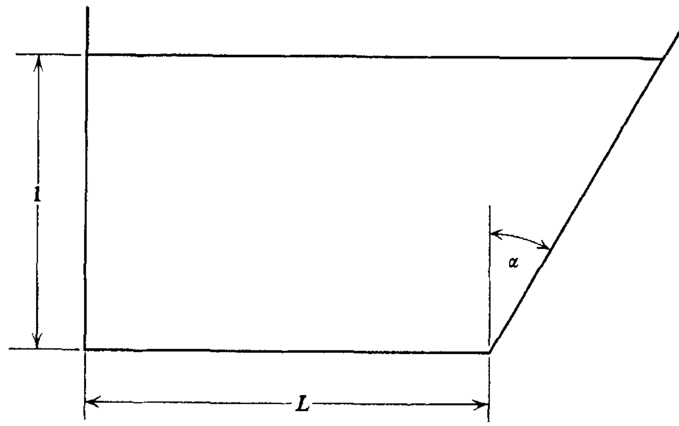

FIGURE 1. Cross-section of channel.

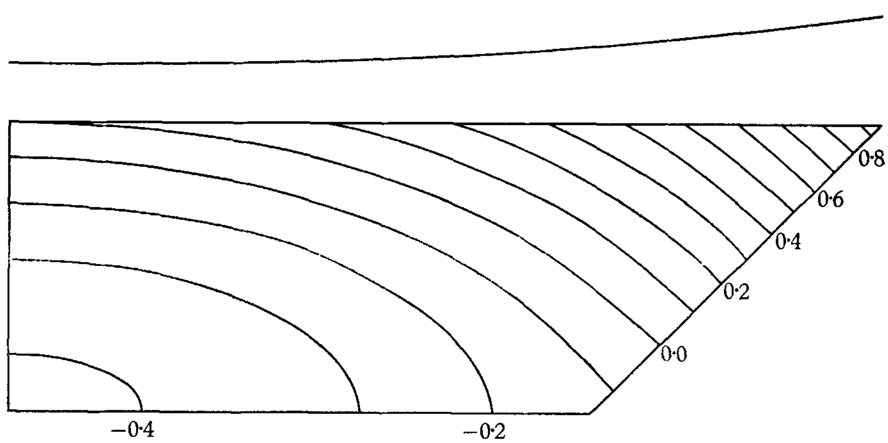

FIGURE 2. The function $\psi(y, z)$ in a trapezium with $L = 2$, $\alpha = 45°$. The lines are drawn at an interval of 0·1. The curve at the top of the diagram shows the variation of ψ along the free surface.

where $$\psi_B = \frac{1}{B_0}\int_{B_0}\psi(y,0)\,dy, \quad \psi_A = \frac{1}{A_0}\iint_{A_0}\psi(y,z)\,dy\,dz$$

and $B(z)$ is the width of the channel at height z. In this case, $A_0 = L+\frac{1}{2}\tan\alpha$, $B(z) = L+z\tan\alpha$, and $\psi_B-\psi_A$ varied between 0·358 and 0·25 for the channels used in the experiments.

3. Experiments

The experiments were conducted in a horizontal channel 28 ft. long, made of steel plate, with a rectangular cross-section 30 in. wide and 15 in. deep. The trapezoidal channels were constructed by fixing sloping plywood boards within

The density structure of the upper ocean was simulated by Turner & Kraus (1967) in an experiment in which continuous mechanical stirring and occasional additions of lighter or denser fluids were applied to the upper boundary of a tank of salt water. The rates at which the density interfaces descended, which equalled the entrainment rates, were observed and hypothesized to be a function of the surface stress or stirring rate, density difference, and depth to the interface. Several seasonal changes of the oceanic thermocline were successfully modelled. The theory used to explain these changes assumed negligible dissipation of kinetic energy by viscosity, an assumption which we believe is questionable.

In the present experiments, the stratification of the stable layer is distributed continuously and nearly uniformly throughout the layer, as in Townsend's and Cromwell's experiments. But unlike Cromwell's experiment the energy input is by heat flux at one boundary, rather than by arbitrary stirring. In contrast to Townsend's experiment, the boundary heat flux is directed into the convective layer and decreases towards the stable interface. The mean thermal structure is consequently non-steady and the convective layer continually deepens. Also, in contrast to Townsend's experiment, the stratification within the stable layer remains essentially constant while the interface rises.

2. Equipment and experimental procedure

The fluid used was distilled water, rather than air, in order to allow both a rather large heating rate and sufficient time to take measurements of the changing thermal structure. The container, as shown in figure 1, was a cylindrical tank with an inside diameter of 54·8 cm and a height of 35·5 cm. Its side walls were insulated by 1·3 cm of sponge rubber on the inside and by 1·3 cm of sponge rubber plus 3·8 cm of fibreglass insulation on the outside. A metal tray containing a circulating water bath at constant temperature was in contact with the upper surface of the distilled water to provide a nearly constant upper boundary temperature. The lower boundary of the container was an aluminium disk of thickness 0·63 cm in thermal contact with copper coils underneath. The coils contained circulating water from another constant-temperature bath. The lower side of the coils was insulated by 1·3 cm of sponge rubber plus 2·5 cm of styrofoam.

For measuring mean temperature at various heights, a long resistance wire of length 510 cm was strung back and forth across a horizontal brass ring support of inner diameter 52 cm and thickness 0·24 cm, as shown in figure 2. The wire was located on the ring in such a manner that the ratio of wire length in a given annulus to the area of that annulus was nearly constant. The wire served as one arm of a d.c. bridge-circuit so that changes in the off-balance voltage were proportional to the variations of horizontally averaged temperature. The platinum-alloy resistance wire had a diameter of 0·005 cm, with its time constant, based on a relative water velocity of 1·0 cm sec^{-1}, estimated to be 4 msec. Located in the same plane as the resistance wire were two thermocouples (denoted in figure 2 by circled $\times$), one in the centre of the container and the other at a radius of 19·3 cm. Their time constant was estimated to be 40 msec. The ring support could be moved vertically at a nearly constant rate of 1·0 cm sec^{-1}

wave amplitudes were usually in the range 0·5–2·0 in. for each depth. A representative selection of photographs of the wave was used for measurement, those chosen being those where the wave appeared to be immediately beneath the reference scale, and where its profile was well defined.

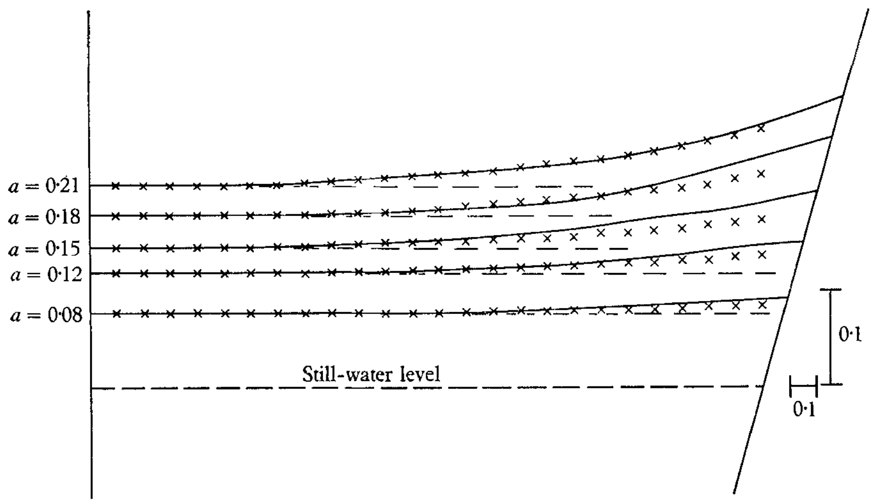

FIGURE 3. The transverse profile of solitary waves in a channel with $L = 1{\cdot}5$, $\alpha = 45°$. The continuous lines show the measured profile; the crosses indicate the theoretical profile. The scales indicate the vertical exaggeration.

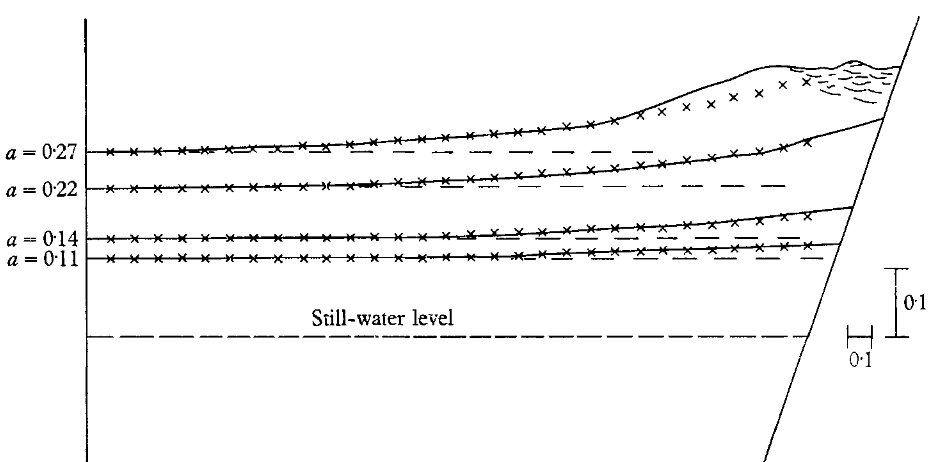

FIGURE 4. The transverse profile of solitary waves in a channel with $L = 2$, $\alpha = 45°$. The continuous lines show the measured profile; the crosses indicate the theoretical profile. The scales indicate the vertical exaggeration.

The experimental profiles were in reasonably good agreement with the theory for all the above six channel cross-sections except for $\alpha = 45°$, $L = 1{\cdot}5$; results for this channel are shown in figure 3. The channel with $\alpha = 45°$, $L = 2$ gave the closest agreement between the theory and experiment and this is shown in figure 4. The continuous lines indicate the measured water surface while the crosses give the theoretical values. In constructing these diagrams some vertical exaggeration has been introduced, as indicated, in order to show the variation more clearly. The amplitudes given are the measured amplitude divided by the maximum depth of the channel.

One possible reason for the anomalously poor results in the channel $\alpha = 45°$, $L = 1{\cdot}5$ is that the phase velocity of asymmetric infinitesimal waves of an appropriate wavelength may be equal to the velocity of the solitary wave. If this is so, the solitary waves may be exciting such waves by resonance and thus causing a divergence between the actual profile of the wave and the theoretical profile obtained by assuming the wave to be steady and alone. In support of this hypothesis it may be noticed that the highest wave shown in figure 3 agrees reasonably well with the theory. A higher solitary wave is shorter and faster than lower ones, whereas shorter infinitesimal waves travel slower.

It was interesting to see how near the theoretical results are to the measured waves when the waves are close to breaking, even though the water surface gradients are so steep that the assumptions on which the theory is based do not hold. The effect of breaking in a trapezoidal channel is not as catastrophic for the wave as it is in a rectangular channel; a wave can continue a long way with little change when it is just breaking near the edge of the channel.

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